<u>**Limit of sequences**</u>

1. **(a)** $$\lim_{n\to\infty}\frac{10000n}{n^2+1}=\lim_{n\to\infty}\frac{\frac{10000}{n}}{1+\frac{1}{n^2}}=\frac{0}{1+0}=0$$

(b) $$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=\lim_{n\to\infty}\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}=\lim_{n\to\infty}\frac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)}=0$$

(c) Since $-1 \le \sin(n!) \le 1 \quad \forall\, n \in N$.

$$-\frac{\sqrt[3]{n^2}}{n+1}\le\frac{\sqrt[3]{n^2}\sin(n!)}{n+1}\le\frac{\sqrt[3]{n^2}}{n+1}\Rightarrow-\lim_{n\to\infty}\frac{\sqrt[3]{n^2}}{n+1}\le\lim_{n\to\infty}\frac{\sqrt[3]{n^2}\sin(n!)}{n+1}\le\lim_{n\to\infty}\frac{\sqrt[3]{n^2}}{n+1}$$

$$\Rightarrow-\lim_{n\to\infty}\frac{\sqrt[3]{\frac{1}{n}}}{1+\frac{1}{n}}\le\lim_{n\to\infty}\frac{\sqrt[3]{n^2}\sin(n!)}{n+1}\le\lim_{n\to\infty}\frac{\sqrt[3]{\frac{1}{n}}}{1+\frac{1}{n}}\Rightarrow 0\le\lim_{n\to\infty}\frac{\sqrt[3]{n^2}\sin(n!)}{n+1}\le 0$$

By sandwich theorem, result follows.

(d) $$\lim_{n\to\infty}\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}=\frac{1}{3}\lim_{n\to\infty}\frac{\left(-\frac{2}{3}\right)^n+1}{\left(-\frac{2}{3}\right)^{n+1}+1}=\frac{1}{3}\left(\frac{0+1}{0+1}\right)=\frac{1}{3}$$

(e) $$\lim_{n\to\infty}\frac{1+a+a^2+\ldots+a^n}{1+b+b^2+\ldots+b^n}=\frac{1-b}{1-a}\lim_{n\to\infty}\frac{(1-a)\left(1+a+a^2+\ldots+a^n\right)}{(1-b)\left(1+b+b^2+\ldots+b^n\right)}=\frac{1-b}{1-a}\lim_{n\to\infty}\frac{1-a^{n+1}}{1-b^{n+1}}=\frac{1-b}{1-a}$$

2. **(a)** $$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right)=\lim_{n\to\infty}\left(\frac{1+2+\ldots+(n-1)}{n^2}\right)=\lim_{n\to\infty}\frac{1}{n^2}\left(\frac{(n-1)n}{2}\right)=\frac{1}{2}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=\frac{1}{2}$$

<u>**or**</u> $$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}+\frac{2}{n}+\ldots+\frac{n-1}{n}\right)=\int_0^1 x\,dx=\frac{x^2}{2}\bigg|_0^1=\frac{1}{2}$$

(b) When n is even, $$L=\lim_{n\to\infty}\frac{1}{n}\left|(1-2)+(3-4)+\ldots+[(n-1)-n]\right|=\lim_{n\to\infty}\frac{1}{n}\left|-\frac{n}{2}\right|=\frac{1}{2}$$

When n is odd, $$L=\lim_{n\to\infty}\frac{1}{n}\left|[n-(n-1)]+\ldots+(3-2)+1\right|=\lim_{n\to\infty}\frac{1}{n}\left|\frac{n+1}{2}\right|=\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)=\frac{1}{2}$$

$\therefore \quad L=\frac{1}{2}$

Note that : $\lim_{n\to\infty}\left[\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\ldots+(-1)^n\frac{n}{n}\right]$ does not exist

since it is equal to $-\frac{1}{2}$ when n is even and $\frac{1}{2}$ when n is odd.

(c) $$L=\lim_{n\to\infty}\frac{1}{n^3}\left(\frac{(n-1)n(2n-1)}{6}\right)=\frac{1}{6}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\cdot1\cdot\left(2-\frac{1}{n}\right)=\frac{1}{3}$$

<u>**or**</u> $$L=\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right]=\int_0^1 x^2\,dx=\frac{x^3}{3}\bigg|_0^1=\frac{1}{3}$$

(d) $L=\lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{i=1}^{n}(2i-1)^2\right)=\lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{i=1}^{n}(4i^2-4i+1)\right)=\lim_{n\to\infty}\frac{1}{n^3}\left[4\times\frac{n(n+1)(2n+1)}{6}-4\times\frac{n(n+1)}{2}+n\right]$

$=\lim_{n\to\infty}\frac{1}{n^3}\left[\frac{n}{3}\left(4n^2-1\right)\right]=\lim_{n\to\infty}\left[\frac{4}{3}-\frac{1}{3n^3}\right]=\frac{4}{3}$

(e) Let $x=\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}$ **(1)**

$2x=1+\frac{3}{2}+\frac{5}{2^2}+\ldots+\frac{2n-1}{2^{n-1}}$ **(2)**

(1) – (2), $x=1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-2}}-\frac{2n-1}{2^n}=1+\frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}}-\frac{2n-1}{2^n}=1+2\left[1-\left(\frac{1}{2}\right)^n\right]-\frac{2n-1}{2^n}$

$L=\lim_{n\to\infty}\left\{1+2\left[1-\left(\frac{1}{2}\right)^n\right]-\frac{2n-1}{2^n}\right\}=3$

(f) $\lim_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}\right]=\lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]=\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]=1$

(g) $L=\lim_{n\to\infty}\left(\sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\cdots\sqrt[2^n]{2}\right)=\lim_{n\to\infty}\left(2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}\ldots 2^{\frac{1}{2^n}}\right)$

$\log_2 L=\lim_{n\to\infty}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}\right]=\frac{1}{2}\lim_{n\to\infty}\left[\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right]=\lim_{n\to\infty}\left[1-\left(\frac{1}{2}\right)^n\right]=1$

$\therefore$ L = 2.

3. **(a)** **<u>Method 1</u>**

To prove P(n) : $2^n > n^2$ $(n \geq 5)$

For P(5), $2^5 = 32 > 25 = 5^2$. $\therefore$ P(5) is true.

Assume P(k) is true for some $k \in N$, $n \geq 5$. (*)

For P(k + 1), $2^{k+1} = 2^k .2 \geq k^2 . 2$, , by (*)

$= k^2 + k^2 = (k + 1)^2 + (k^2 - 2k - 1) = (k + 1)^2 + [(k - 1)^2 - 2] \geq (k + 1)^2 + [5^2 - 2]$

$\geq (k + 1)^2$.

$\therefore$ P(k + 1) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n\in N$, $n \geq 5$.

Therefore $\forall\, n\in N$, $n \geq 5$, $0<\frac{n}{2^n}<\frac{n}{n^2}=\frac{1}{n}$, by P(n) above.

Since $\lim_{n\to\infty}\frac{1}{n}=0$, $0\leq\lim_{n\to\infty}\frac{n}{2^n}\leq\lim_{n\to\infty}\frac{1}{n}=0$. By sandwich theorem, result follows.

<u>Method 2</u>

For $n > 2$, $2^n=(1+1)^n>1+n+\frac{n(n-1)}{2}\Rightarrow 0<\frac{n}{2^n}<\frac{n}{1+n+\frac{n(n-1)}{2}}$

Take $n\to\infty$ and apply Sandwich theorem, result follows.

3. **(b)** $0 < \frac{2^n}{n!} = \frac{2}{1}\cdot\frac{2}{2}\cdot\left(\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{2}{5}\cdot\ldots\cdot\frac{2}{n}\right) < 2\cdot 1\cdot\left(\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{2}{3}\right) = 2\left(\frac{2}{3}\right)^{n-2}$

$0 \le \lim_{n\to\infty}\frac{2^n}{n!} \le \lim_{n\to\infty} 2\left(\frac{2}{3}\right)^{n-2} = 0$ By sandwich theorem, result follows.

(c) First we like to prove that $\lim_{n\to\infty}\frac{n}{a^n} = 0$

Let $a = 1 + h$, since $a > 1$, $\therefore$ $h > 0$.

$$0 < \frac{n}{a^n} = \frac{n}{(1+h)^n} = \frac{n}{1 + nh + \frac{n(n-1)}{2}h^2 + \ldots + h^n} < \frac{n}{\frac{n(n-1)}{2}h^2} = \frac{2}{(n-1)h^2}$$

$0 < \lim_{n\to\infty}\frac{n}{a^n} \le \lim_{n\to\infty}\frac{2}{(n-1)h^2} = 0$ $\therefore \lim_{n\to\infty}\frac{n}{a^n} = 0$, by Sandwich Theorem.

Now we like to prove that $\lim_{n\to\infty}\frac{n^k}{a^n} = 0$, where k = constant , $a > 1$.

We can assume that $k > 0$, since the identity is obvious true where $k \le 0$.

Let $a = b^k$, since $a > 1$, $b > 1$.

$$\lim_{n\to\infty}\frac{n^k}{a^n} = \lim_{n\to\infty}\frac{n^k}{(b^k)^n} = \left[\lim_{n\to\infty}\frac{n}{b^n}\right]^k = 0^k = 0$$

(d) Let k be a fixed natural number greater than a, then for $n > k$,

$$0 < \frac{a^n}{n!} = \left(\frac{a}{1}\cdot\frac{a}{2}\cdot\frac{a}{3}\cdots\frac{a}{k}\right)\cdot\left(\frac{a}{k+1}\cdot\frac{a}{k+2}\cdots\frac{a}{n}\right) < \frac{a^k}{k!}\cdot\frac{a}{n}$$

$\lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{a^n}{n!} \le \lim_{n\to\infty}\frac{a^k}{k!}\cdot\frac{a}{n}$ $\Rightarrow \lim_{n\to\infty}\frac{a^n}{n!} = 0$, by Sandwich Theorem.

(e) $|q| < 1$. $\therefore$ $\frac{1}{|q|} > 1$, let $\frac{1}{|q|} = 1 + h$. $\therefore$ $h > 0$ and $|q| = \frac{1}{1+h}$.

$$0 < |nq^n| = \frac{n}{(1+h)^n} = \frac{n}{1 + nh + \frac{n(n-1)}{2}h^2 + \ldots + h^n} < \frac{n}{\frac{n(n-1)}{2}h^2} = \frac{2}{(n-1)h^2}$$

$0 \le \lim_{n\to\infty}|nq^n| \le \lim_{n\to\infty}\frac{2}{(n-1)h^2} = 0$ $\Rightarrow$ $\lim_{n\to\infty}|nq^n| = 0$, by Sandwich Theorem.

(f) **(i)** If $a > 1$, put $a_n = \sqrt[n]{a} - 1$, then $a_n > 0$.

By Bernoulli's inequality, $1 + na_n \le (1 + a_n)^n = a$. Hence $0 < a_n \le \frac{a-1}{n}$.

Taking limit $n \to\infty$ and using Sandwich theorem, result follows.

(ii) If $a = 1$, the result is obvious.

(iii) If $0 < a < 1$, then put $b = 1/a$.

Since $b > 1$, by (i), $\lim_{n\to\infty}\sqrt[n]{b} = 1$ $\Rightarrow \lim_{n\to\infty}\frac{1}{\sqrt[n]{a}} = 1$ $\Rightarrow \frac{1}{\lim_{n\to\infty}\sqrt[n]{a}} = 1$ $\Rightarrow \lim_{n\to\infty}\sqrt[n]{a} = 1$

(g) If $a_n = \sqrt[n]{n}$, let us write $a_n = 1 + b_n$, where $b_n > 0$.

Now, $n = a_n^n = (1 + b_n)^n = 1 + nb_n + \frac{n(n-1)}{2}b_n^2 + \ldots + b_n^n > \frac{n(n-1)}{2}b_n^2$.

Thus, $0 < b_n < \sqrt{\frac{2}{n-1}}$ $\Rightarrow 0 \le \lim_{n\to\infty} b_n \le \lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$ $\Rightarrow \lim_{n\to\infty} b_n = 0$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}(1+b_n) = 1$.

(h) $0 < \sqrt[n]{\frac{1}{n!}} = \sqrt[2n]{\frac{1}{(n!)^2}} = \sqrt[2n]{\left[\frac{1}{n}\right]\left[\frac{1}{n(n-1)}\right]\left[\frac{1}{(n-1)(n-2)}\right]\ldots\left[\frac{1}{3\cdot 2}\right]\left[\frac{1}{2\cdot 1}\right]}$

$< \sqrt[2]{\frac{1}{n}\left\{\left[\frac{1}{n}\right]+\left[\frac{1}{n(n-1)}\right]+\left[\frac{1}{(n-1)(n-2)}\right]+\ldots+\left[\frac{1}{3\cdot 2}\right]+\left[\frac{1}{2\cdot 1}\right]\right\}}$ (A.M. > G.M.)

$< \sqrt[2]{\frac{1}{n}\left\{\left[\frac{1}{n}\right]+\left[\frac{1}{n-1}-\frac{1}{n}\right]+\left[\frac{1}{n-2}-\frac{1}{n-1}\right]+\ldots+\left[\frac{1}{2}-\frac{1}{3}\right]+\left[1-\frac{1}{2}\right]\right\}} = \sqrt{\frac{1}{n}}$

$\therefore \quad 0 \le \lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}} \le \lim_{n\to\infty}\sqrt{\frac{1}{n}} = 0$ $\Rightarrow \lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}} = 0$, by Sandwich Theorem.

Note : 3(f), (g), (h) can be proved by the result of No.7.

4. $\lim_{n\to\infty} a_n = a \Rightarrow \forall \varepsilon > 0,\ \exists N \in \mathbf{N}$ s.t. $n > N \Rightarrow |a_n - a| < \varepsilon$

$\forall \varepsilon > 0,\ \exists N \in \mathbf{N}$ s.t. $n > N \Rightarrow \left||a_n| - |a|\right| \le |a_n - a| < \varepsilon$ $\quad \therefore \lim_{n\to\infty}|a_n| = |a|$

The converse is not true. Counterexample : Consider $a_n = (-1)^n$.

a_n is an oscillating sequence and $\lim_{n\to\infty}(-1)^n$ does not exist.

But $\lim_{n\to\infty}\left|(-1)^n\right| = \lim_{n\to\infty} 1 = 1$ exist.

5. Let $a_n = b_n + a$, we must prove that $\lim_{n\to\infty} b_n = 0 \Rightarrow \lim_{n\to\infty}\frac{b_1+b_2+\ldots+b_n}{n} = 0$.

Now, $\frac{b_1+b_2+\ldots+b_n}{n} = \frac{b_1+b_2+\ldots+b_p}{n} + \frac{b_{p+1}+b_{p+2}+\ldots+b_n}{n}$

So that, $\left|\frac{b_1+b_2+\ldots+b_n}{n}\right| \le \frac{|b_1+b_2+\ldots+b_p|}{n} + \frac{|b_{p+1}|+|b_{p+2}|+\ldots+|b_n|}{n}$ (1)

Since $\lim_{n\to\infty} b_n = 0$, we can choose p so that for $n > p$, $|b_n| < \varepsilon/2$.

Then $\frac{|b_{p+1}|+|b_{p+2}|+\ldots+|b_n|}{n} < \frac{\frac{\varepsilon}{2}+\frac{\varepsilon}{2}+\ldots+\frac{\varepsilon}{2}}{n} = \frac{(n-p)\frac{\varepsilon}{2}}{n} < \frac{\varepsilon}{2}$ (2)

After choosing p we can choose N so that $\forall n > N > p$, $\frac{|b_1+b_2+\ldots+b_p|}{n} < \frac{\varepsilon}{2}$ (3)

Then (2) and (3) , (1) becomes: $\left|\frac{b_1+b_2+\ldots+b_n}{n}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ for $n > N$.

Thus proving the result.

6. $\lim_{n\to\infty} a_n = a \Rightarrow \forall \varepsilon > 0,\ \exists n \in N$ s.t. $n > N \Rightarrow |a_n - a| < a\varepsilon$ and $a_n > 0$ (since $a > 0$)

$\therefore \quad |\log_e a_n - \log_e a| = \left|\log_e \frac{a_n}{a}\right| = \left|\log_e\left(1+\frac{a_n - a}{a}\right)\right| < \left|\frac{a_n - a}{a}\right| = \frac{a\varepsilon}{a} = \varepsilon$ $\quad \therefore \lim_{n\to\infty}\log_e a_n = \log_e a$

(**Note :** $\log_e(1 + x) < x$ for $x > -1 , x \ne 0$.)

7. By (6), $\lim_{n\to\infty} a_n = a \quad (a > 0)$, $\quad \lim_{n\to\infty} \log_e a_n = \log_e a$

By (5), $\lim_{n\to\infty} \frac{\log_e a_1 + \log_e a_2 + \ldots + \log_e a_n}{n} = \log_e a$

$$\Rightarrow \lim_{n\to\infty} \log_e \sqrt[n]{a_1 a_2 \ldots a_n} = \log_e a$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a_1 a_2 \ldots a_n} = a$$

8. Put $b_1 = a_1$, $\quad b_2 = \frac{a_2}{a_1}$, $\quad \ldots$, $\quad b_n = \frac{a_n}{a_{n-1}}$

Then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = b$ (say)

By (7), $\lim_{n\to\infty} \sqrt[n]{b_1 b_2 \ldots b_n} = b \quad \Rightarrow \lim_{n\to\infty} \sqrt[n]{a_1 \frac{a_2}{a_1} \ldots \frac{a_n}{a_{n-1}}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \quad \Rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

9. Put $a_n = \frac{(2n)!}{(n!)^2}$, then $a_{n+1} = \frac{[2(n+1)]!}{[(n+1)!]^2}$.

$$\frac{a_{n+1}}{a_n} = \frac{(2n+1)(2n+2)}{(n+1)^2} = \frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2}. \qquad \therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4 \ .$$

By No. 4, $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{(2n)!}{(n!)^2}\right)^{1/n} = 4$.

10. First we like to prove that $y_n > y_{n+1}$. $\quad y_n = \left(1+\frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1}$

$$= \frac{1}{\left(\frac{n}{n+1}\right)^{n+1}} = \frac{1}{\left(1-\frac{1}{n+1}\right)^{n+1}} = \frac{1}{1\times\left(1-\frac{1}{n+1}\right)^{n+1}} < \frac{1}{\left[\frac{1+(n+1)\left(1-\frac{1}{n+1}\right)}{1+(n+1)}\right]^{n+2}} \quad , \quad \text{A.M.} > \text{G..M.}$$

$$= \frac{1}{\left(\frac{n+1}{n+2}\right)^{n+2}} = \left(\frac{n+2}{n+1}\right)^{n+2} = \left(1+\frac{1}{n+1}\right)^{n+2} = y_{n+1}$$

$\therefore$ y_n is monotonic decreasing . It is also bounded below since $y_n > 0$.

$\therefore$ $\lim_{n\to\infty} y_n$ exists.

$$\therefore \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = e \ .$$

11. Let $x_n = \left(1+\frac{1}{n}\right)^n$, $y_n = \sum_{r=0}^{n}\frac{1}{r!}$

$$x_n = \left(1+\frac{1}{n}\right)^n = 1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\left(\frac{1}{n}\right)^2+\ldots+\binom{n}{k}\left(\frac{1}{n}\right)^k+\ldots+\binom{n}{n}\left(\frac{1}{n}\right)^n$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$$

Since $k < n$, $x_n < 2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)$

Letting $n\to\infty$, we have: $e \ge 2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{k!} = y_k$

Clearly, $x_n < y_n \le e$ and so as $\lim_{n\to\infty} y_n$ exists and is equal to e, by Sandwich theorem.

12. $a_{n+1}-a_n = \frac{1}{n}-\ln\left(1+\frac{1}{n}\right)$, $b_{n+1}-b_n = \frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)$

Since for $n < x < n + 1$, we have $\frac{1}{n+1} < \frac{1}{x} < \frac{1}{n}$.

$$\therefore \int_n^{n+1}\frac{dx}{n+1} < \int_n^{n+1}\frac{dx}{x} < \int_n^{n+1}\frac{dx}{n} \quad \Rightarrow \frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

$\therefore a_{n+1}-a_n > 0$, $b_{n+1}-b_n < 0$.

$\{a_n\}$ is monotonic increasing and $\{b_n\}$ is monotonic decreasing.

Furthermore, $a_n < b_n$.

Hence $a_1 = 0$ is the lower bound of b_n, $b_n > a_n > a_{n-1} > \ldots > a_1$.

and $b_1 = 1$ is the upper bound of a_n, $a_n < b_n < \ldots < b_1$.

By Monotone Bounded Convergence theorem, $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ exists.

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty}\frac{1}{n} = 0 \qquad \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

13. $0 < a_n = \frac{3}{4}\times\frac{5}{6}\times\frac{7}{8}\times\ldots\times\frac{2n-1}{2n}\times\frac{1}{2n+2} < \frac{1}{2n+2}$

Taking limit as $n\to\infty$, we have, $0 \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty}\frac{1}{2n+2} = 0$ $\therefore \lim_{n\to\infty} a_n = 0$.

14. It can be seen that $a_n > 0 \quad \forall\, n \in \mathbf{N}$.

Now, $a_1 > 2$, $a_2 = \frac{6}{1+a_1} < \frac{6}{1+2} = 2$.

But $a_{n+1} - 2 = \frac{6}{1+a_n} - 2 = \frac{4-2a_n}{1+a_n} = -\frac{2(a_n-2)}{1+a_n}$ (*)

If $a_n > 2$, then $a_n - 2 > 0$, from (*) since $1 + a_n > 0$, we have $a_{n+1} - 2 > 0$ or $a_{n+1} > 2$.

Similarly, if $a_n < 2$, then $a_{n+1} < 2$.

By the Principle of Mathematical Induction, $a_{2k-1} > 2$, $a_{2k} < 2 \quad \forall\, k \in \mathbf{N}$.

Now, $a_{n+2} - a_n = \dfrac{6}{1+a_n} - a_n = \dfrac{6}{1+\dfrac{6}{1+a_n}} - a_n = \dfrac{6(1+a_n)}{7+a_n} - a_n = -\dfrac{{a_n}^2 + a_n - 6}{7+a_n} = -\dfrac{(a_n - 2)(a_n + 3)}{7+a_n}$

If $n = 2k - 1$, then $a_{2k+1} - a_{2k-1} = -\dfrac{(a_{2k-1} - 2)(a_{2k-1} + 3)}{7 + a_{2k-1}} < 0$, since $a_{2k-1} > 2$.

If $n = 2k$, then $a_{2k+2} - a_{2k} = -\dfrac{(a_{2k} - 2)(a_{2k} + 3)}{7 + a_{2k}} > 0$, since $a_{2k} < 2$.

$\{ a_{2k-1} \}$ is decreasing and is bounded below by 2.

$\{ a_{2k} \}$ is increasing and is bounded above by 2.

$\therefore$ $\{ a_{2k-1} \}$ and $\{ a_{2k} \}$ have limits.

Since $a_{2k+2} = \dfrac{6(1+a_{2k})}{7+a_{2k}}$. Take limit $k \to \infty$, putting $a = \lim\limits_{k\to\infty} a_{2k} = \lim\limits_{k\to\infty} a_{2k+2}$, we get

$a = \dfrac{6(1+a)}{7+a}$, $a^2 + 7a - 6 = 0$, $(a + 3)(a - 2) = 0$, $\therefore$ $a = 2$ ($a = -3$ is rejected as $a > 0$)

$\therefore$ The sequence $\{ a_{2k} \}$ is convergent to 2.

Since $a_{2k+1} = \dfrac{6(1+a_{2k-1})}{7+a_{2k-1}}$. Take limit $k \to \infty$, putting $b = \lim\limits_{k\to\infty} a_{2k-1} = \lim\limits_{k\to\infty} a_{2k+1}$, we get

$b = \dfrac{6(1+b)}{7+b}$, $\therefore$ $b = 2$ $\therefore$ The sequence $\{ a_{2k-1} \}$ is also convergent to 2.

Since limit of a sequence is unique if exists, $\lim\limits_{n\to\infty} a_n = a = b = 2$.

15. It can be seen easily that $a_n > 0$ $\forall$ $n \in \mathbf{N}$.

$a_{n+1} - 1 = \dfrac{1}{2}\left(a_n + \dfrac{1}{a_n}\right) - 1 = \dfrac{{a_n}^2 + 1}{2a_n} - 1 = \dfrac{{a_n}^2 + 1 - 2a_n}{2a_n} = \dfrac{(a_n - 1)^2}{2a_n} > 0$

$\Rightarrow a_{n+1} > 1$ $\quad \Rightarrow a_n > 1$ $\quad \forall n \in N$ $\quad$ (1)

Now, $a_{n+1} - a_n = \dfrac{1}{2}\left(a_n + \dfrac{1}{a_n}\right) - a_n = \dfrac{{a_n}^2 + 1}{2a_n} - a_n = \dfrac{{a_n}^2 + 1 - 2{a_n}^2}{2a_n} = \dfrac{1 - {a_n}^2}{2a_n} = \dfrac{(1 - a_n)(1 + a_n)}{2a_n} < 0$, by (1).

$\therefore$ $a_{n+1} - a_n < 0$, $a_{n+1} < a_n$.

$\therefore$ $\{ a_n \}$ is decreasing and is bounded below by 1.

$\therefore$ $\lim\limits_{n\to\infty} a_n$ exists. Let $L = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$

Since $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{1}{a_n}\right)$, take $n \to \infty$, we have $L = \dfrac{1}{2}\left(L + \dfrac{1}{L}\right) = \dfrac{L^2 + 1}{2L}$

$\therefore$ $2L^2 = L^2 + 1$

$L^2 = 1$

Since $a_n > 0$, $L > 0$, $\therefore$ $L = 1$.

16. **(i)** a_n is strictly increasing

Let $P(n)$ be the proposition : $a_n < a_{n+1}$.

For $P(1)$, $a_1 = \sqrt{6}$, $a_2 = \sqrt{6+\sqrt{6}} > \sqrt{6+0} = \sqrt{6} = a_1$

Assume $P(k)$ is true for some $k \in \mathbf{N}$. i.e. $a_k < a_{k+1}$ …. (1)

(1) $\Rightarrow$ $6 + a_k < 6 + a_{k+1}$ $\Rightarrow$ $a_{k+1} = \sqrt{6+a_k} < \sqrt{6+a_{k+1}} = a_{k+2}$

$\therefore$ $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true $\forall n \in \mathbf{N}$.

(ii) a_n is bounded above.

Since $a_n > 0$, $a_n = \sqrt{6+a_{n-1}} \Rightarrow a_n^2 = 6 + a_{n-1} \Rightarrow a_n = \dfrac{6}{a_n} + \dfrac{a_{n-1}}{a_n} < \dfrac{6}{a_n} + 1$, a_n is increasing.

But $a_n > a_{n-1} > \ldots > a_1 = \sqrt{6}$ $\Rightarrow \dfrac{6}{a_n} < \dfrac{6}{\sqrt{6}}$ $\therefore$ $a_n < \dfrac{6}{\sqrt{6}} + 1$

(iii) By Bounded Monotone Theorem, $\lim\limits_{n\to\infty} a_n$ exists. Let $L = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$

By taking $n \to \infty$, $a_n = \sqrt{6+a_{n-1}} \Rightarrow L = \sqrt{6+L} \Rightarrow L^2 - L + 6 = 0 \Rightarrow (L-3)(L-2) = 0$

Since $L > 0$, $\therefore L = 3$.

17. **(i)** $u_{n+1} - u_n = \dfrac{2(n+1)-3}{(n+1)+2} - \dfrac{2n-3}{n+2} = \dfrac{2n+5}{n+3} - \dfrac{2n-3}{n+2} = \dfrac{(2n+5)(n+2)-(2n-3)(n+3)}{(n+2)(n+3)} = \dfrac{1}{(n+2)(n+3)} > 0$

$\therefore u_{n+1} > u_n$ and u_n is a montonic increasing sequence.

(ii) $u_n = \dfrac{2n-3}{n+2} = 2 - \dfrac{1}{n+2} < 2$ $\therefore$ u_n is bounded above.

By Monotone Bound Theorem, the limit $\lim\limits_{n\to\infty} u_n$ exists. $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{2+3/n}{1+2/n} = 2$.

18. **(a)** S_n diverges. It is bounded.

(b) S_n converges. $S_n \to 0$ as $n \to \infty$.

(c) S_n diverges. It is unbounded. It is not true that $S_n \to \infty$ as $n \to \infty$.

(d) S_n converges. $S_n \to 1$ as $n \to \infty$.

(e) S_n diverges. It is bounded.

(f) S_n converges. $S_n \to 0$ as $n \to \infty$.

(g) S_n diverges. It is unbounded. It is true that $S_n \to \infty$ as $n \to \infty$.

(h) S_n diverges. It is unbounded. It is true that $S_n \to \pm\infty$ as $n \to \infty$.

However, it is not true that $S_n \to +\infty$ as $n \to \infty$.

19. **(a)** No, take $a_n = 1/n$, then $a_n \to 0$ as $n \to \infty$.

$$\sum_{n=0}^{\infty} a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

Since the last series diverges, Σa_n diverges.

(b) Yes, put $s_n = a_1 + a_2 + \dots + a_n$.

Σa_n converges, then $\lim_{n\to\infty} s_n$ exists and is equal to L , say.

$\forall \varepsilon > 0,\ \exists N \in \mathbf{N}$ s.t. $n > N \Rightarrow |s_n - L| < \varepsilon/2$ and $|s_{n-1} - L| < \varepsilon/2$

Then , for this N, n > N,

$|a_n - 0| = |s_n - s_{n-1}| = |(s_n - L) - (L - s_{n-1})| < |s_n - L| + |L - s_{n-1}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\therefore$ $\lim_{n\to\infty} a_n = 0$.

(c) If $\Sigma | a_n |$ is convergent, then by Cauchy definition:

$\forall \varepsilon > 0,\ \exists N \in \mathbf{N}$ s.t. $n > N, \forall p \in \mathbf{N} \Rightarrow |a_{n+1}| + |a_{n+2}| + \dots + |a_{n+p}| < \varepsilon$

$\therefore$ $0 < |a_{n+1} + a_{n+2} + \dots + a_n| < |a_{n+1}| + |a_{n+2}| + \dots + |a_{n+p}| < \varepsilon$

$\therefore$ Σa_n is convergent.

(d) No, take $a_n = (-1)^n \dfrac{1}{n}$, then Σa_n is convergent.

Proof : $|s_{n+p} - s_n| = \dfrac{1}{n+1} - \dfrac{1}{n+2} + \dfrac{1}{n+3} - \dots + (-1)^{p-1}\dfrac{1}{n+p} < \dfrac{1}{n+1} < \dfrac{1}{n}$

$\forall \varepsilon > 0$, take $N = [1/\varepsilon]$, then if $n > N$, $|s_{n+p} - s_n| < \dfrac{1}{n} < \varepsilon$ holds.

But $\Sigma a_n = \Sigma(1/n)$ is not convergent.

$|s_{n+p} - s_n| = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+p} > \dfrac{1}{n+p} + \dfrac{1}{n+p} + \dots + \dfrac{1}{n+p} > \dfrac{p}{n+p}$

Take $p = n$, $|s_{2n} - s_n| > \dfrac{1}{2}$. If we take $\varepsilon = \dfrac{1}{2}$, no matter how large is N, we cannot have

$|s_{2n} - s_n| < \dfrac{1}{2} = \varepsilon$.

20. $a_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \dots + \dfrac{1}{\sqrt{n^2+n}} < \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+1}} + \dots + \dfrac{1}{\sqrt{n^2+1}} < \dfrac{n}{\sqrt{n^2+1}}$

$a_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \dots + \dfrac{1}{\sqrt{n^2+n}} > \dfrac{1}{\sqrt{n^2+n}} + \dfrac{1}{\sqrt{n^2+n}} + \dots + \dfrac{1}{\sqrt{n^2+n}} > \dfrac{n}{\sqrt{n^2+n}}$

$\therefore$ $\dfrac{n}{\sqrt{n^2+n}} < a_n < \dfrac{n}{\sqrt{n^2+1}}$ …. (1)

Taking limit as $n \to \infty$.

$\lim_{n\to\infty} \dfrac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \dfrac{1}{\sqrt{1+1/n}} = 1$ and $\lim_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \dfrac{1}{\sqrt{1+1/n^2}} = 1$.

By Sandwich theorem, $\lim_{n\to\infty} a_n = 1$.

21. **(a)** $x_1 = 2^{1/2}, \dots, x_n = (2x_{n-1})^{1/2}$. First we like to prove that $P(n) : x_n < 2\ \ \forall n \in \mathbf{N}$.

For P(1), $x_1 = \sqrt{2} < 2$ $\therefore$ P(1) is true.

Assume that P(k-1) is true for some $\forall k \in \mathbf{N}$, i.e. $x_{k-1} < 2$.

For P(k), $x_k = (2x_{k-1})^{1/2} < (2\times 2)^{1/2} = 2$

By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbf{N}$.

Now, $x_n = (2x_{n-1})^{1/2} \Rightarrow x_n{}^2 = 2x_{n-1} \Rightarrow \dfrac{x_n}{x_{n-1}} = \dfrac{2}{x_n} = \dfrac{2}{2} = 1$, since $x_n > 0$ and $x_n < 2$.

$\therefore$ x_n is an increasing function and is bounded above by 2.

By the Monotone Bound Theorem, $\lim\limits_{n\to\infty} x_n$ exists and is equal to L (say).

Since by taking $n\to\infty$, $x_n = (2x_{n-1})^{1/2} \Rightarrow L = (2L)^{1/2} \Rightarrow L^2 = 2L \Rightarrow L = 0,2$

Since $x_n > 0$, L = 2.

(b) $x_1 = c^{1/2}, \ldots, x_n = (c + x_{n-1})^{1/2}$, $c > 0$.

(i) $x_n > 0$ $\forall n \in \mathbf{N}$. (obviously true, or prove by math. induction)

(ii) Claim: $x_n < x_{n+1}$ $\forall n \in \mathbf{N}$. Use M.I. $x_1 = \sqrt{c} < \sqrt{c+\sqrt{c}} = x_2$

$$x_{n-1} < x_n \Rightarrow c + x_{n-1} < c + x_n \Rightarrow \sqrt{c + x_{n-1}} < \sqrt{c + x_n} \Rightarrow x_n < x_{n+1}$$

(iii) Let $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{n+1} = L$,then $x_n = (c + x_{n-1})^{1/2} \Rightarrow L = \sqrt{c+L} \Rightarrow L = \dfrac{1+\sqrt{1+4c}}{2}$, $L > 0$.

22. (i) $u_{n+1} = \dfrac{6(1+u_n)}{7+u_n}$, $u_1 = c > 2$

$$u_{n+1} - 2 = \frac{6(1+u_n)}{7+u_n} - 2 = \frac{6 + 6u_n - 14 - 2u_n}{7+u_n} = \frac{4u_n - 8}{7+u_n} = \frac{4(u_n - 2)}{7+u_n} \quad \ldots(*)$$

It is clear that $u_n > 0$, $\forall n \in \mathbf{N}$. Hence by (*), $u_n > 2 \Rightarrow u_{n+1} > 2$.

Now, $u_{n+1} - u_n = \dfrac{6(1+u_n)}{7+u_n} - u_n = -\dfrac{(u_n - 2)(u_n + 3)}{7+u_n} < 0$.

$\therefore$ $u_{n+1} < u_n$ and u_n is monotonic decreasing.

(ii) Similar to (i) , we can use mathematical induction to show that $u_n < 2$ $\forall n \in \mathbf{N}$.

$$u_{n+1} - u_n = \frac{6(1+u_n)}{7+u_n} - u_n = -\frac{(u_n - 2)(u_n + 3)}{7+u_n} > 0$$

$\therefore$ $u_{n+1} > u_n$ and u_n is monotonic increasing.

u_n is bounded below in (i) since $u_n > 0$ $\forall n \in \mathbf{N}$.

u_n is bounded above in (ii) since $u_n < 2$ $\forall n \in \mathbf{N}$.

$\therefore$ Limit exists in both the above cases. Let $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{n+1} = L$, then from

$$u_{n+1} = \frac{6(1+u_n)}{7+u_n} \Rightarrow L = \frac{6(1+L)}{7+L} \Rightarrow (L-2)(L+3) = 0 \Rightarrow L = 2\,.$$

L = -3 is rejected since $u_n > 0 \Rightarrow L > 0$.

23. $u_n = \sqrt[n]{a^n + b^n}$, $0 < b \le a \Rightarrow \dfrac{b}{a} \le 1 \Rightarrow \left(\dfrac{b}{a}\right)^n \le 1$,

$$a = \sqrt[n]{a^n} \le \sqrt[n]{a^n + b^n} = a\sqrt[n]{(b/a)^n + 1^n} \le \sqrt[n]{1^n + 1^n} = a(2)^{1/n}$$

Taking limit as $n\to\infty$ and using Sandwich theorem, $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \sqrt[n]{a^n + b^n} = a$.

24. We like to use Math. Induction to show that P(n) : $a < x_n < b$ is true $\forall n \in \mathbf{N}$.

For P(1), $a < x_1 = h < b$. $\therefore$ P(1) is true.

Assume P(k) is true for some $n \in \mathbf{N}$. $a < x_k < b$ …. (1)

For P(k + 1), $x_{k+1} = x_k^2 + c > a^2 + c = a$ (since $a^2 - a + c = 0$)

$x_{k+1} = x_k^2 + c < b^2 + c = b$ (since $b^2 - b + c = 0$)

$\therefore$ $a < x_{k+1+} < b$ and P(k + 1) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbf{N}$.

We like to use Math. Induction to show that P(n) : $x_{n+1} < x_n$ is true $\forall n \in \mathbf{N}$.

For P(1), $x_1 = h$, $x_2 = x_1^2 + c = h^2 + c = (h^2 - h + c) + h < h$ (since $h^2 - h + c < 0$)

(Note that a, b are roots of $x^2 - x + c = 0 \Rightarrow a + b = 1$ and $ab = c$

$h^2 - h + c = h^2 - (a + b) h + ab = (h - a)(h - b) < 0$ since $a < h < b \Leftrightarrow h - a > 0$ and $h - b < 0$)

$\therefore$ P(1) is true.

Assume P(k) is true for some $n \in \mathbf{N}$. $x_{k+1} < x_k$ for some $n \in \mathbf{N}$.

For P(k + 1), $x_{k+1} < x_k \Rightarrow x_{k+1}^2 < x_k^2 \Rightarrow x_{k+1}^2 + c < x_k^2 + c \Rightarrow x_{k+2} < x_{k+1}$ $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbf{N}$.

Since x_n is bounded and is monotonic decreasing, limit exists. Let $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = L$.

By $x_{n+1} = x_n^2 + c$, as $n \to \infty$, $L = L^2 + c \Rightarrow L^2 - L + c = 0 \Rightarrow L = a$ or b (L = b is rejected)

25. **(a)** We like to prove that $\forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$, $n > N \Rightarrow \left|\frac{n+1}{n^3+1}\right| < \varepsilon$.

But $\left|\frac{n+1}{n^3+1}\right| < \varepsilon \Leftarrow \frac{n+1}{n^3+1} < \varepsilon \Leftarrow \frac{1}{n^2+n+1} < \varepsilon \Leftarrow \frac{1}{n^2} < \varepsilon \Leftarrow n^2 > \frac{1}{\varepsilon}$

Take $N = \left[\frac{1}{\varepsilon}\right]$, then $n > N \Rightarrow \left|\frac{n+1}{n^3+1}\right| < \varepsilon$.

(b) We like to prove that $\forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$, $n > N \Rightarrow \left|\frac{\sin n}{n}\right| < \varepsilon$.

But $\left|\frac{\sin n}{n}\right| < \varepsilon$ is true if $\left|\frac{1}{n}\right| < \varepsilon$, since $|\sin n| < 1$.

or if $1/n < \varepsilon$ or if $n > 1/\varepsilon$.

Take $N = \left[\frac{1}{\varepsilon}\right]$, then $n > N \Rightarrow \left|\frac{\sin n}{n}\right| < \varepsilon$.

(c) We like to prove that $\forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$, $n > N \Rightarrow \left|\frac{n+(-1)^n}{n^2-1}\right| < \varepsilon$

$$\left|\frac{n+(-1)^n}{n^2-1}\right| < \varepsilon \Leftarrow \frac{n+(-1)^n}{n^2-1} < \varepsilon \quad (\because n \geq 1) \Leftarrow \frac{n+1}{n^2-1} < \varepsilon (\because (-1)^n \leq 1) \Leftarrow \frac{1}{n-1} < \varepsilon \Leftarrow n > \frac{1}{\varepsilon} - 1$$

Take $N = \left[\frac{1}{\varepsilon} - 1\right]$, then $n > N \Rightarrow \left|\frac{n+(-1)^n}{n^2-1}\right| < \varepsilon$.

(d) We like to prove that $\forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$, $\forall p > 0, n > N \Rightarrow |u_{n+p} - u_p| < \varepsilon$

But $|u_{n+p} - u_p| = \left|\frac{1}{(p+1)n} - \frac{1}{(p+2)n} + \ldots + (-1)^{n+1}\frac{1}{(n+p)^2}\right| < \varepsilon$ is true if $\frac{1}{(p+1)n} < \varepsilon \Leftarrow \frac{1}{n} < \varepsilon \Leftarrow n > \frac{1}{\varepsilon}$. Take $N = \left[\frac{1}{\varepsilon}\right]$, then $n > N \Rightarrow |u_{n+p} - u_p| < \varepsilon$.

(e) We like to prove that $\forall \varepsilon > 0, \quad \exists N(\varepsilon) > 0, \quad n > N \Rightarrow \left|(n+1)^{1/2} - n^{1/2}\right| < \varepsilon$. But:

$$\left|(n+1)^{1/2} - n^{1/2}\right| < \varepsilon \Leftarrow (n+1)^{1/2} - n^{1/2} = \frac{1}{(n+1)^{1/2} + n^{1/2}} < \varepsilon \Leftarrow (n+1)^{1/2} + n^{1/2} > \frac{1}{\varepsilon} \Leftarrow n^{1/2} + n^{1/2} > \frac{1}{\varepsilon}$$

$$\Leftarrow n^{1/2} + n^{1/2} > \frac{1}{\varepsilon} \Leftarrow 2n^{1/2} > \frac{1}{\varepsilon} \Leftarrow n > \sqrt{\frac{1}{2\varepsilon}} \qquad \therefore \text{Take} \quad N = \left[\sqrt{\frac{1}{2\varepsilon}}\right] ,$$

we have $\forall \varepsilon > 0, \quad \exists N(\varepsilon) > 0, \quad n > N \Rightarrow \left|(n+1)^{1/2} - n^{1/2}\right| < \varepsilon$.

26. **(a)** When $n = 2k$, $\lim_{k\to\infty} \frac{2k-1}{2k} = 1 \quad \therefore r_{2k} \to 1$.

When $n = 2k - 1$, $\lim_{k\to\infty} \frac{(2k-1)+1}{2k-1} = 1 \quad \therefore r_{2k-1} \to 1. \qquad \therefore r_n \to 1.$

(b) 3 **(c)** 3/2

(d) $\lim_{n\to\infty} n^{1/2} = \infty$. We like to prove $\forall M > 0, \exists N > 0, \quad n > N \Rightarrow |n^{1/2}| > M$

But $|n^{1/2}| > M$ is true if $n > M^2$. Take $N = M^2, n > N \Rightarrow |n^{1/2}| > M$.

(e) $\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) = \infty$. See solution in No. 19 (d).

(f) $\lim_{n\to\infty} \log_{10} n = \infty$ We like to prove that $\forall M > 0, \exists N > 0, \quad n > N \Rightarrow |\log_{10} n| > M$

But $|\log_{10} n| > M$ is true if $n > 10^M$. Take $N = 10^M$, then $n > N \Rightarrow |\log_{10} n| > M$.

27. $\lim_{n\to\infty} x_n = 0. \quad \forall \varepsilon > 0, \quad \exists N(\varepsilon) > 0, \quad n > N \Rightarrow |x_n| < \varepsilon$.

$\therefore \quad \forall\ 1/\varepsilon > 0, \quad \exists N(\varepsilon) > 0, \quad n > N \Rightarrow |1/x_n| > \varepsilon$ since $|x_n| \neq 0$. Take $M = [1/\varepsilon]$, then

$\forall\ M > 0, \quad \exists N(\varepsilon) > 0, \quad n > N \Rightarrow |1/x_n| > M \quad . \quad \therefore \quad \lim_{n\to\infty} \frac{1}{x_n} = \infty$.

28. Yes. Proof : Let $x_n \to A$, $y_n \to B$. x_n has limit $\Rightarrow x_n$ is bounded $\Leftrightarrow |x_n| < M$.

$|x_ny_n - AB| = |x_ny_n - x_n B + x_n B - AB| \le |x_ny_n - x_n B| + |x_n B - AB| = |x_n|\ |y_n - B| + |B||x_n - A|$

$x_n \to A , \quad \forall \varepsilon > 0, \quad \exists N(\varepsilon) > 0, \quad n > N \Rightarrow |x_n - A| < \varepsilon$

$y_n \to B, \quad \forall \varepsilon > 0, \quad \exists N(\varepsilon) > 0, \quad n > N \Rightarrow |y_n - B| < \varepsilon$

$\therefore |x_ny_n - AB| < M\varepsilon + B\varepsilon = \varepsilon\,(M + |B|) \qquad \therefore \quad \lim_{n\to\infty} x_n y_n = AB$.

The converse is not true. Take $x_ny_n = 1$, $x_n = n$, $y_n = 1/n$.

29. **(a)** $\mathbf{0} <$ $\frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2} < \frac{1}{n^2} + \frac{1}{n^2} + \ldots + \frac{1}{n^2} < \frac{n}{n^2} = \frac{1}{n}$

Take $n \to \infty$, $0 \le \frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2} \le \lim_{n\to\infty} \frac{1}{n} = 0 \qquad \therefore \quad \lim_{n\to\infty} \frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2} = 0$

(b) $$\lim_{n\to\infty}\frac{1+\frac{1}{2}+\ldots+\frac{1}{2^n}}{1+\frac{1}{4}+\ldots+\frac{1}{4^n}}=\lim_{n\to\infty}\frac{\left(1-\left(\frac{1}{2}\right)^n\right)/\left(1-\frac{1}{2}\right)}{\left(1-\left(\frac{1}{4}\right)^n\right)/\left(1-\frac{1}{4}\right)}=\frac{3}{2}\lim_{n\to\infty}\frac{1-\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{4}\right)^n}=\frac{3}{2}$$

(c) (sin n!) is bounded and $\lim_{n\to\infty}\frac{n-1}{n^2-1}=\lim_{n\to\infty}\frac{1}{n+1}=0 \quad \therefore \lim_{n\to\infty}(\sin n!)\left(\frac{n-1}{n^2-1}\right)^{18}=0$

$$\lim_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{(n-1)n}\right]=\lim_{n\to\infty}\left\{\left[\frac{1}{1}-\frac{1}{2}\right]+\left[\frac{1}{2}-\frac{1}{3}\right]+\ldots+\left[\frac{1}{n-1}-\frac{1}{n+1}\right]\right\}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$$

$$\lim_{n\to\infty}\frac{2n^2+1}{n^2-1}=\lim_{n\to\infty}\frac{2+\left(1/n^2\right)}{1-\left(1/n^2\right)}=2\ ,$$

$$\therefore \quad \lim_{n\to\infty}\left\{(\sin n!)\left[\frac{n-1}{n^2+1}\right]^{18}-\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{(n-1)n}\right]\left[\frac{2n^2+1}{n^2-1}\right]\right\}=-2$$

(d) $$\lim_{n\to\infty}\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}=\lim_{n\to\infty}\frac{3^n\left[(-2/3)^n+1\right]}{3^{n+1}\left[(-2/3)^{n+1}+1\right]}=\frac{1}{3}\lim_{n\to\infty}\frac{(-2/3)^n+1}{(-2/3)^{n+1}+1}=\frac{1}{3}$$

30. Put $a = 1 + b$, given that $a > 1 \quad \therefore b > 0$

$$a^n=(1+b)^n>1+nb+\frac{n(n-1)}{2}b^2 \Rightarrow \frac{1}{a^n}<\frac{1}{1+nb+n(n-1)b^2/2}\Rightarrow 0<\frac{n}{a^n}<\frac{n}{1+nb+n(n-1)b^2/2}$$

Take $n\to\infty$, $\quad 0\le\lim_{n\to\infty}\frac{n}{a^n}\le\lim_{n\to\infty}\frac{n}{1+nb+n(n-1)b^2/2}=0 \qquad \therefore \quad \lim_{n\to\infty}\frac{n}{a^n}=0$.

$$0<\frac{n^5}{2^n}=\frac{n^5}{(1+1)^n}<\frac{n^5}{1+n+n(n-1)/2+\ldots+n(n-1)\ldots(n-5)/6!}$$ where $n\ge 6$.

Take $n\to\infty$ and use Squeeze Theorem, $\lim_{n\to\infty}\frac{n^5}{2^n}=0$.

31. (a) $$\lim_{n\to\infty}\frac{\sin\frac{5}{n^2}}{\tan\frac{1}{n^2}}=\lim_{n\to\infty}5\times\frac{\sin\frac{5}{n^2}}{\frac{5}{n^2}}\Bigg/\frac{\tan\frac{1}{n^2}}{\frac{1}{n^2}}=\lim_{x\to 0}5\times\frac{\sin x}{x}\Big/\frac{\tan x}{x}=5$$

(b) $$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}\times n=\lim_{x\to 0}\frac{\sin x}{x}\lim_{n\to\infty}n=1\times\infty=\infty$$

(c) $$\lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{\sqrt{n}}\right)}{\sqrt{n}}\underset{x=\frac{2}{\sqrt{n}}}{=}\frac{1}{2}\lim_{x\to 0}[x\ln(1+x)]=\frac{1}{2}\lim_{x\to 0}x\lim_{x\to 0}[\ln(1+x)]=\frac{1}{2}\times 0\times 0=0$$

(d) $$\lim_{n\to\infty}\frac{1+2+\ldots+n}{n^2+1}=\lim_{n\to\infty}\frac{n(n+1)/2}{n^2+1}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2+n}{n^2+1}=\frac{1}{2}\lim_{n\to\infty}\frac{1+(1/n)}{1+\left(1/n^2\right)}=\frac{1}{2}$$

(e) $$\lim_{n\to\infty}\frac{\sqrt{n^2+1}+\sqrt{n}-\sqrt{n^2-1}}{n+1}=\lim_{n\to\infty}\frac{\sqrt{1+\left(1/n^2\right)}+\sqrt{1/n}-\sqrt{1-\left(1/n^2\right)}}{1+(1/n)}=0$$

32. Given that $x_1 = 1$, $x_2 = 2$ and $x_n = \sqrt{x_{n-1}x_{n-2}}$ (n > 2)

$$\frac{x_n}{x_{n-1}} = \frac{\sqrt{x_{n-1}x_{n-2}}}{x_{n-1}} = \sqrt{\frac{x_{n-2}}{x_{n-1}}} = \left(\frac{x_{n-1}}{x_{n-2}}\right)^{-1/2} = \left(\frac{x_{n-2}}{x_{n-3}}\right)^{(-1/2)^2} = \dots = \left(\frac{x_2}{x_1}\right)^{(-1/2)^{n-2}} = 2^{(-1/2)^{n-2}}$$

$$x_n = 2^{(-1/2)^{n-2}} x_{n-1} = 2^{(-1/2)^{n-2}} 2^{(-1/2)^{n-3}} x_{n-1} = \dots = 2\exp\left[\left(-\frac{1}{2}\right)^{n-2} + \left(-\frac{1}{2}\right)^{n-2} + \dots + \left(-\frac{1}{2}\right)^{0}\right] x_1$$

$$= 2\exp\left[\frac{1-(-1/2)^{n-1}}{1-(-1/2)}\right] = 2\exp\left[\frac{3}{2}\left(1-\left(-\frac{1}{2}\right)^n\right)\right].$$

$\therefore \quad \lim_{n\to\infty} x_n = 2^{3/2} = 2\sqrt{2}$.

33. $a > b > 0$, $a_1 = \frac{1}{2}(a+b)$, $b_1 = \frac{2ab}{a+b}$, $a_n = \frac{1}{2}(a_{n-1}+b_{n-1})$, $b_n = \frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}}$

$a_nb_n = a_{n-1}b_{n-1} = \dots = a_1b_1 = ab$. Obviously $a_n > 0, b_n > 0$, since $a > b > 0$.

$$a_n = \frac{1}{2}(a_{n-1}+b_{n-1}) \underset{A.M.\ge G.M.}{\ge} \sqrt{a_nb_n} = \sqrt{ab} \qquad \Rightarrow a_n^2 \ge ab \qquad \dots. \quad (1)$$

$$b_n = \frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}} \underset{H.M.\le G.M.}{\le} \sqrt{a_nb_n} = \sqrt{ab} \qquad \Rightarrow b_n^2 \le ab \qquad \dots. \quad (2)$$

$$a_n - a_{n-1} = a_n - \frac{1}{2}(a_{n-1}+b_{n-1}) = a_n - \frac{1}{2}\left(a_{n-1} + \frac{ab}{a_{n-1}}\right) = \frac{1}{2}\left(\frac{ab - a_{n-1}^2}{a_{n-1}}\right) \le 0 \quad \text{, by (1)}$$

$$b_n - b_{n-1} = \frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}} - b_{n-1} = \frac{a_{n-1}b_{n-1} - b_{n-1}^2}{a_{n-1}+b_{n-1}} \ge 0$$

$\sqrt{ab} \le a_n \le a_{n-1}$, $\sqrt{ab} \ge b_n \ge b_{n-1}$.

a_n is montonic decreasing and is bounded below. b_n is montonic increasing and is bounded above.

Let $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = L$, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n-1} = M$.

$$a_n = \frac{1}{2}(a_{n-1}+b_{n-1}), \qquad b_n = \frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}} \Rightarrow L = \frac{L+M}{2},\ M = \frac{2LM}{L+M} \Rightarrow L = M \quad (L, M \ne 0)$$

$$a_n = \frac{1}{2}(a_{n-1}+b_{n-1}) = \frac{1}{2}\left(a_{n-1} + \frac{ab}{a_{n-1}}\right) \Rightarrow L = \frac{1}{2}\left(L + \frac{ab}{L}\right) \Rightarrow L = \sqrt{ab} \quad \text{(negative root rejected)}$$

34. (i) $$x_{n+1} - x_n = \frac{3(1+x_n)}{3+x_n} - x_n = \frac{3-x_n^2}{3+x_n} \qquad \dots. \quad (1)$$

$$x_n - x_{n-1} = \frac{3-x_{n-1}^2}{3+x_{n-1}} \qquad \dots. \quad (2)$$

Now, $$3 - x_n^2 = 3 - \left[\frac{3(1+x_{n-1})}{3+x_{n-1}}\right]^2 = \frac{6(3-x_{n-1}^2)}{(3+x_{n-1})^2} \qquad \dots. \quad (3)$$

From (3), $3-x_n^2$, $3-x_{n-1}^2$ are of the same sign.

From (1) and (2), $x_{n+1} - x_n$ and $x_n - x_{n-1}$ are of the same sign.

$\therefore$ x_n is monotonic.

(ii) $$\left|x_{n+1} - \sqrt{3}\right| = \left|\frac{3(1+x_n)}{3+x_n} - \sqrt{3}\right| = \left|\frac{3+3x_n - 3\sqrt{3} - \sqrt{3}x_n}{3+x_n}\right| = \left|\frac{(3-\sqrt{3})(x_n-\sqrt{3})}{3+x_n}\right| = \left|\frac{3-\sqrt{3}}{3+x_n}\right|\left|x_n - \sqrt{3}\right|$$

$< \left|\frac{3}{3+x_n}\right| \left|x_n - \sqrt{3}\right| = k\left|x_n - \sqrt{3}\right|$ where $k = \left|\frac{3}{3+x_n}\right|$, $0 < k < 1$, since $x_n > 0$.

$\left|x_{n+1} - \sqrt{3}\right| < k\left|x_n - \sqrt{3}\right| < k^2\left|x_{n-1} - \sqrt{3}\right| < \ldots < k^n\left|x_1 - \sqrt{3}\right|$

$\therefore$ $\sqrt{3} - k^n\left|x_1 - \sqrt{3}\right| < x_{n+1} < \sqrt{3} + k^n\left|x_1 - \sqrt{3}\right|$ and x_{n+1} (hence x_n) is bounded.

By (i) x_n is monotonic, $\therefore$ x_n has a limit.

Let $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{n+1} = L$, then from $x_{n+1} = \frac{3(1+x_n)}{3+x_n}$, $L = \frac{3(1+L)}{3+L}$, $L = 3$ (-ve root rejected)

35. **(i)** $u_{n+1} - A = \frac{u_n^{\ 2} + A^2}{2u_n} - A = \frac{(u_n - A)^2}{2u_n} \geq 0$ since $u_n \geq 0$. $\therefore$ $u_{n+1} \geq A$. (u_n is bounded.)

$u_{n+1} - u_n = \frac{u_n^{\ 2} + A^2}{2u_n} - u_n = \frac{A^2 - u_n^{\ 2}}{2u_n}$, but $u_n \geq A \Rightarrow u_n^{\ 2} \geq A^2$, since $u_n \geq 0$, $A > 0$.

$\therefore$ $u_{n+1} \leq u_n$. (u_n is montonic decreasing.)

(ii) $d_{n+1} = \frac{u_{n+1} - A}{u_{n+1} + A} = \frac{\frac{u_n^{\ 2} + A^2}{2u_n} - A}{\frac{u_n^{\ 2} + A^2}{2u_n} + A} = \frac{u_n^{\ 2} + A^2 - 2u_n A}{u_n^{\ 2} + A^2 + -2u_n A} = \left(\frac{u_n - A}{u_n + A}\right)^2 = d_n^{\ 2}$

(iii) By (i), limit exists.

By (ii), $d_{n+1} = d_n^{\ 2} = (d_{n-1})^2 = \ldots = d_1^{\ 2^n}$ …. (1)

Since $0 < A \leq u_1$, $d_1 = \frac{u_1 - A}{u_1 + A}$, $0 < d_1 < 1$.

As $n \to \infty$, by (1), $\lim\limits_{n\to\infty} d_1^{\ 2^n} = 0 \Rightarrow \lim\limits_{n\to\infty} d_n = \lim\limits_{n\to\infty} \frac{u_n - A}{u_n + A} = 0 \Rightarrow \lim\limits_{n\to\infty} u_n = A$.

Let $A = \sqrt{11}$, $A^2 = 11$, $u_1 = 4$ $\Rightarrow 0 < A < u_1$.

$u_2 = \frac{u_1^{\ 2} + A^2}{2u_1} = \frac{16+11}{2\times 4} = 3.375$,

$u_3 = \frac{u_2^{\ 2} + A^2}{2u_2} = \frac{3.375^2 + 11}{2\times 3.375} \approx 3.3173$

$u_4 = \frac{u_3^{\ 2} + A^2}{2u_3} = \frac{3.3173^2 + 11}{2\times 3.3173} \approx 3.3166$

$u_5 = \frac{u_4^{\ 2} + A^2}{2u_4} = \frac{3.3166^2 + 11}{2\times 3.3166} \approx 3.3166$

$\therefore$ $\sqrt{11} \approx 3.317$ (correct to 3 decimal places)